Functional equations for double zeta-functions

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Abstract

As the first step of research on functional equations for multiple zeta-functions, we present a candidate of the functional equation for a class of two variable double zeta-functions of the Hurwitz–Lerch type, which includes the classical Euler sum as a special case.

1. Introduction

Let $u_1, \ldots, u_r$ be complex variables. The $r$-variable Euler–Zagier sum is a kind of multiple zeta-function defined by the series

$$
\zeta_r(u_1, \ldots, u_r) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-u_1} (m_1 + m_2)^{-u_2} \cdots (m_1 + \cdots + m_r)^{-u_r},
$$

which is convergent absolutely when $\Re(u_{r-k+1} + \cdots + u_r) > k (1 \leq k \leq r)$. The analytic continuation of (1.1) as an $r$-variable meromorphic function has recently been established (Arakawa and Kaneko [2], Zhao [11], Akiyama, Egami and Tanigawa [1] and the author [8, 9]). A problem that follows naturally is to search for the functional equation(s), this has not yet been successful.

The aim of this paper is to propose a candidate of the functional equation for the simplest case $r = 2$, that is the classical Euler sum. Hereafter we write it as

$$
\zeta_2(u, v) = \sum_{m=1}^{\infty} m^{-u} \sum_{n=1}^{\infty} (m+n)^{-v}.
$$

Let $\Gamma(u), \zeta(u)$ be the gamma function and the Riemann zeta-function, respectively. Define

$$
g(u, v) = \zeta_2(u, v) - \frac{\Gamma(1 - u)}{\Gamma(v)} \Gamma(u + v - 1) \zeta(u + v - 1) \cdot
$$

We use the notation $\sigma_\ell(k) = \sum_{d|k} d^\ell$. Let

$$
\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xy} x^{a-1} (1+y)^{c-a-1} dy
$$

be the confluent hypergeometric function, where $\Re a > 0, -\pi < \phi < \pi, |\phi + \arg x| < \pi/2$. Then our functional equation can be formulated, in terms of $g(u, v)$, as follows:
Theorem 1. We have
\[ g(u, v) = \frac{g(1-v, 1-u)}{i^{u+v-1} \Gamma(v)} + 2i \sin\left(\frac{\pi}{2}(u+v-1)\right) F_+(u, v), \] (1.4)
where \( i = \sqrt{-1} = \exp(\pi i/2) \) and \( F_+(u, v) \) is the series defined by
\[ F_+(u, v) = \sum_{k=1}^{\infty} \sigma_{u+v-1}(k) \Psi(v, u + v; 2\pi ik). \] (1.5)

The series (1.5) is convergent only in the region \( \Re u < 0, \Re v > 1 \), but it can be continued meromorphically to the whole \( \mathbb{C}^2 \) space.

Remark. The function \( F_+(u, v) \) itself satisfies a nice functional equation. See Proposition 2 in Section 3.

In the following sections we will prove the functional equation of a more general double zeta-function, which will show the duality more clearly. The main result (Theorem 2) will be stated in the last section.

2. Double Hurwitz–Lerch zeta-functions

When \( r = 1 \), the series (1.1) is nothing but the Riemann zeta-function, whose functional equation is well known. A classical generalization of the Riemann zeta-function is the Hurwitz zeta-function \( \zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \), where \( \alpha > 0 \). The functional equation of \( \zeta(s, \alpha) \) is of the form
\[ \zeta(s, \alpha) = \frac{\Gamma(1-s)}{\iota(2\pi)^{1-s}} \left( e^{\pi i s/2} \phi(1-s, \alpha) - e^{-\pi i s/2} \phi(1-s, -\alpha) \right), \] (2.1)
where \( \phi(s, \alpha) = \sum_{n=1}^{\infty} \exp(2\pi in\alpha)n^{-s} \) is the Lerch zeta-function. (See [10, 2.17.3].) More generally, define the Hurwitz–Lerch zeta-function by
\[ \zeta(s, \alpha, \beta) = \sum_{n=0}^{\infty} \frac{\exp(2\pi in\beta)}{(n + \alpha)^s}. \] (2.2)

When \( 0 < \beta < 1 \), the functional equation of \( \zeta(s, \alpha, \beta) \) is given by
\begin{align*}
\zeta(s, \alpha, \beta) &= \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left( e^{\pi i (1-s)/2} e^{-2\pi i \alpha \beta} \zeta(1-s, \beta, -\alpha)ight.
\left. + e^{-\pi i (1-s)/2} e^{2\pi i \alpha (1-\beta)} \zeta(1-s, 1-\beta, \alpha) \right) \tag{2.3}
\end{align*}
which expresses the transparent duality between, not only \( s \) and \( 1-s \), but also \( \alpha \) and \( \beta \) [4, section 1.8], see also [5, theorem 2∗].

In [7], the author introduced the following generalized double zeta-function:
\[ \zeta_2(u, v; \alpha, w) = \sum_{m=0}^{\infty} (\alpha + m)^{-u} \sum_{n=1}^{\infty} (\alpha + m + nw)^{-v}, \] (2.4)
where \( \alpha > 0 \) and \( w > 0 \). This double series was further studied in [8]. The reason for introducing the weight \( w \) is to include the Barnes double zeta-function as a special case \( u = 0 \). The above (2.4) is a kind of two-variable generalization of the Hurwitz zeta-function. In view of (2.3), however, it is better to introduce a more general double series with some exponential factor, when we consider the subject of
functional equations. In the present paper we consider the following two-variable
doubleseriesoftheHurwitz–Lerchtype:

\[ \zeta_2(u, v; \alpha, \beta, w) = \sum_{m=0}^{\infty} (\alpha + m)^{-u} \sum_{n=1}^{\infty} e^{2\pi i n \beta} (\alpha + m + nw)^{-v}, \]  

(2.5)

where \( 0 < \alpha \leq 1, 0 \leq \beta \leq 1, \) and \( w > 0. \) This series is convergent absolutely when \( \Re u > 1, \Re v > 1. \)

In this section we prove a certain infinite series expression of \( \zeta_2(u, v; \alpha, \beta, w). \) The argument is similar to that developed in [7], hence we only give a brief sketch.

First, similarly to [7, 3-4], we can show

\[ \zeta_2(u, v; \alpha, \beta, w) = \frac{1}{\Gamma(u)\Gamma(v)} \int_{0}^{\infty} \frac{y^{v-1}}{e^{2\pi i \beta e^{w y}} - 1} \int_{0}^{\infty} \frac{x^{u-1}}{e^{x+y} - 1} dx dy. \]  

(2.6)

The right-hand side is convergent when \( \Re u > 0, \Re v > 1, \) and \( \Re (u + v) > 2. \) Let

\[ h(z; \alpha) = \frac{e^{(1-\alpha)z}}{e^z - 1} - \frac{1}{z}, \]

and divide the right-hand side of (2.6) as

\[ \frac{1}{\Gamma(u)\Gamma(v)} \int_{0}^{\infty} \frac{y^{v-1}}{e^{2\pi i \beta e^{w y}} - 1} \int_{0}^{\infty} \frac{x^{u-1}}{e^{x+y} - 1} dx dy \]

\[ + \frac{1}{\Gamma(u)\Gamma(v)} \int_{0}^{\infty} \frac{y^{v-1}}{e^{2\pi i \beta e^{w y}} - 1} \int_{0}^{\infty} h(x+y; \alpha) x^{u-1} dx dy \]

\[ = g_0(u, v; \alpha, \beta, w) + g(u, v; \alpha, \beta, w), \]

(2.7)

say. We can show

\[ g_0(u, v; \alpha, \beta, w) = \frac{\Gamma(1-u)}{\Gamma(v)} \Gamma(u + v - 1) \phi(u + v - 1, \beta) w^{1-u-v}. \]

Let \( C \) be the contour which consists of the half-line on the positive real axis from
infinity to a small positive number, a small circle counterclockwise round the origin,
and the other half-line on the positive real axis back to infinity. Deforming the path
to the contour \( C, \) we have

\[ g(u, v; \alpha, \beta, w) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \]

\[ \times \int_{C} \frac{y^{v-1}}{e^{2\pi i \beta e^{w y}} - 1} \int_{C} h(x+y; \alpha) x^{u-1} dx dy. \]  

(2.8)

Since the right-hand side of (2.8) is convergent absolutely for \( \Re u < 1 \) and any \( v, \) in
the same region we obtain

\[ \zeta_2(u, v; \alpha, \beta, w) = \frac{\Gamma(1-u)}{\Gamma(v)} \Gamma(u + v - 1) \phi(u + v - 1, \beta) w^{1-u-v} \]

\[ + g(u, v; \alpha, \beta, w). \]  

(2.9)

This is a generalization of [7, 3-11].
Since $0 < \alpha \leq 1$, we can apply the method in [7, section 5]. Let $\Re u < 0, \Re v > 1$. Changing the path of the inner integral on the right-hand side of (2·8) by a circle of large radius $R$, counting the residues of relevant poles, and letting $R \to \infty$, we obtain

$$g(u, v; \alpha, \beta, w) = \frac{-2\pi i}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \sum_{n \neq 0} e^{-2\pi i n \alpha} I_n,$$  

(2·10)

where

$$I_n = \int_c y^{u-1} e^{-2\pi i \beta e^w y} (-y + 2\pi i n)^{u-1} dy.$$  

(2·11)

The integral $I_n$ is similar to $I_\eta(\tau)$ appearing in [6, section 5]. Applying the method in [6, pp. 35–37], we obtain the expression of $g(u, v; \alpha, \beta, w)$ analogous to [7, 5·4] which can be written, in terms of confluent hypergeometric functions (1·3), as follows. In the statement, we use the notation

$$F_\pm(u, v; \alpha, \beta, w) = \sum_{k=1}^{\infty} \sigma_{u+v-1}(k; \alpha, \beta) \Psi(v, u+v; \pm 2\pi i k w),$$  

(2·12)

where

$$\sigma_\ell(k; \alpha, \beta) = \sum_{d|k} e^{2\pi i \alpha} e^{2\pi i (k/d) \beta} d^\ell.$$  

(2·13)

**Proposition 1.** We have

$$g(u, v; \alpha, \beta, w) = (2\pi)^{u+v-1} \Gamma(1 - u) \times \{e^{\pi i (u+v-1)/2} F_+(u, v; \alpha, \beta, w) + e^{-\pi i (1-u-v)/2} F_-(u, v; -\alpha, \beta, w)\}$$  

(2·14)

in the region $\Re u < 0, \Re v > 1$.

This is a generalization of [7, 5·5].

**Remark.** Since $\sigma_\ell(k; \alpha, \beta)$ and $F_\pm(u, v; \alpha, \beta, w)$ are periodic (of period 1) with respect to $\alpha$ and also with respect to $\beta$, we can extend the definition of $g(u, v; \alpha, \beta, w)$ for any real $\alpha$ and $\beta$ by this periodicity.

### 3. Properties of $F_\pm(u, v; \alpha, \beta, w)$

In this section we discuss the basic properties of the functions $F_\pm(u, v; \alpha, \beta, w)$. First recall well-known properties of $\Psi(a, c; x)$. They are the transformation formula

$$\Psi(a, c; x) = x^{1-c} \Psi(a - c + 1, 2 - c; x),$$  

(3·1)

[3, formula 6·5(6)], and the asymptotic expansion

$$\Psi(a, c; x) = \sum_{k=0}^{N-1} \frac{(-1)^k (a - c + 1)k(a)k}{k!} x^{-a-k} + \rho_N(a, c; x),$$  

(3·2)

[3, formula 6·13·1(1)], where $N$ is an arbitrary non-negative integer, $(a)_k = \Gamma(a + k)/\Gamma(a)$ and $\rho_N(a, c; x)$ is the remainder term which can be explicitly
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written as

\[ \rho_N(a, c; x) = \frac{(-1)^N(a - c + 1)_N}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-xy} y^{a+N-1} \]
\[ \times \int_0^1 \frac{(1 - \tau)^{N-1}}{(N - 1)!} (1 + \tau y)^{c-a-N-1} d\tau dy. \]  

(3·3)

First we assume \( \Re u < 0 \) and \( \Re v > 1 \). From (2·12) and (3·1) we have

\[ F_\pm(u, v; \alpha, \beta, w) = \sum_{k=1}^{\infty} \sigma_{u+v-1}(k; \alpha, \beta)(\pm 2\pi i w)^{1-u-v} \]
\[ \times \Psi(1 - u, 2 - u - v; \pm 2\pi i w). \]

(3·4)

Applying the fact

\[ \sigma_{u+v-1}(k; \alpha, \beta)k^{1-u-v} = \sigma_{1-u-v}(k; \beta, \alpha) \]

(3·5)

to (3·4), we find

\[ F_\pm(u, v; \alpha, \beta, w) = (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^{\infty} \sigma_{1-u-v}(k; \beta, \alpha) \]
\[ \times \Psi(1 - u, 2 - u - v; \pm 2\pi i w) \]
\[ = (\pm 2\pi i w)^{1-u-v} F_\pm(1 - v, 1 - u; \beta, \alpha, w), \]

(3·6)

which is the functional equation for \( F_\pm(u, v; \alpha, \beta, w) \).

Applying (3·2) to (3·4), we obtain

\[ F_\pm(u, v; \alpha, \beta, w) \]
\[ = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} (v)^j (1 - u)^j (\pm 2\pi i w)^{-v-j} \sum_{k=1}^{\infty} \frac{\sigma_{u+v-1}(k; \alpha, \beta)}{k^{j+v}} \]
\[ + (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^{\infty} \sigma_{u+v-1}(k; \alpha, \beta)k^{1-u-v} \]
\[ \times \rho_N(1 - u, 2 - u - v; \pm 2\pi i w) \]
\[ = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} (v)^j (1 - u)^j (\pm 2\pi i w)^{-v-j} \]
\[ \times \phi(j - u + 1, \alpha)\phi(j + v, \beta) \]
\[ + (\pm 2\pi i w)^{1-u-v} \sum_{k=1}^{\infty} \sigma_{u+v-1}(k; \alpha, \beta)k^{1-u-v} \]
\[ \times \rho_N(1 - u, 2 - u - v; \pm 2\pi i w). \]

(3·7)

Noting

\[ |\sigma_{u+v-1}(k; \alpha, \beta)| \leq \sigma_{\Re(u+v)-1}(k), \]

similarly to [7, 6·3] we find that the second term on the right-hand side of (3·7) is convergent absolutely for \( \Re u < N \) and \( \Re v > -N + 1 \). Therefore, since \( N \) is arbitrary, (3·7) implies that \( F_\pm(u, v; \alpha, \beta, w) \) can be continued meromorphically to the whole \( \mathbb{C}^2 \) space. Summarizing the above argument, we now obtain
Proposition 2. The function $F_\pm(u, v; \alpha, \beta, w)$ can be continued meromorphically to the whole $(u, v)$ space, and satisfies the functional equation

$$F_\pm(1 - v, 1 - u; \beta, \alpha, w) = (\pm 2\pi i w)^{u+v-1}F_\pm(u, v; \alpha, \beta, w).$$  \hfill (3.8)

4. The main result

Now it is easy to complete the proof of our main result, that is the functional equation of $\zeta(u, v; \alpha, \beta, w)$, which is written in terms of $g(u, v; \alpha, \beta, w)$ as follows.

Theorem 2. Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, and $w > 0$. Then we have the functional equation

$$\frac{g(u, v; \alpha, \beta, w)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v, 1-u; 1-\beta, 1-\alpha, w)}{(iw)^{u+v-1}\Gamma(v)} + e^{\pi i (u+v-1/2)}F_+(u, v; \alpha, \beta, w) - e^{\pi i(1-u-v)/2}F_-(u, v; 1-\alpha, 1-\beta, w).$$  \hfill (4.1)

When $\beta = 1$, we define $g(1-v, 1-u; 0, 1-\alpha, w) = g(1-v, 1-u; 1, 1-\alpha, w)$ (see the remark at the end of Section 2). In particular, when $\alpha = \beta = 1$, from (4.1) we obtain

$$\frac{g(u, v; 1, 1, w)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v, 1-u; 1, 1, w)}{(iw)^{u+v-1}\Gamma(v)} + 2i \sin \left(\frac{\pi}{2}(u + v - 1)\right)F_+(u, v; 1, 1, w).$$  \hfill (4.2)

Theorem 1 is a special case of this formula.

Proof of Theorem 2. First of all we note that, in view of Proposition 2, we now know that (2.14) gives the meromorphic continuation of $g(u, v; \alpha, \beta, w)$ to the whole $(u, v)$ space. Changing $u, v, \alpha, \beta$ by $1 - v, 1 - u, 1 - \beta, 1 - \alpha$ in (2.14), we have

$$g(1-v, 1-u; 1-\beta, 1-\alpha, w) = (2\pi)^{1-u-v}\Gamma(v)\left\{ e^{\pi i(1-u-v)/2}F_+ \times (1-v, 1-u; 1-\beta, 1-\alpha, w) + e^{\pi i(u+v-1)/2}F_-(1-v, 1-u; \beta-1, 1-\alpha, w) \right\}. $$

Substituting (3.8) into the right-hand side of the above, we obtain

$$g(1-v, 1-u; 1-\beta, 1-\alpha, w) = \Gamma(v)w^{u+v-1}\{F_+(u, v; 1-\alpha, 1-\beta, w) + F_-(u, v; 1-\alpha, \beta-1, w)\}$$

$$= \Gamma(v)w^{u+v-1}\{F_+(u, v; 1-\alpha, 1-\beta, w) + F_-(u, v; -\alpha, \beta, w)\}. $$ \hfill (4.3)

From (2.14) and (4.3), by eliminating the terms $F_-(u, v; -\alpha, \beta, w)$, we arrive at the desired assertion.

It is highly desirable to extend our result to multiple zeta-functions (1.1) for $r \geq 3$. We can show the integral expression

$$\zeta_r(u_1, \ldots, u_r) = \frac{1}{\Gamma(u_1) \cdots \Gamma(u_r)} \int_0^\infty \cdots \int_0^\infty x_1^{u_1-1} \cdots x_r^{u_r-1} \times \frac{1}{e^{x_1+\cdots+x_r} - 1} \cdots \frac{1}{e^{x_r} - 1} \, dx_1 \cdots dx_r.$$ \hfill (4.4)

(cf. [2, theorem 3(ii)]). Similarly to (2.7), we can divide this into $2^{r-1}$ integrals. By using the beta integral formula, we can see that the simplest integral among them, corresponding to $g_0(u, v; \alpha, \beta, w)$ in (2.7), can be expressed as a product of $\Gamma$-factors and a zeta-function. However it does not seem easy to find transformation formulas for remaining integrals.
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